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LETTER TO THE EDITOR

Symmetries and first integrals for dissipative systems

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Abstract. The connection between symmetries, time independent and time dependent first integrals for autonomous systems of first-order ordinary differential equations is discussed. Several new results which have been obtained by applying a REDUCE package are reported.

Let $\boldsymbol{x}=(x_1,\dots x_n)$ be n functions depending on the variable t and let

$$\Omega_\nu(t,\boldsymbol{x})\equiv \dot{x}_\nu-\omega_\nu(\boldsymbol{x})=0 \qquad (\nu=1,\dots,n) \tag{1}$$

be a system of n autonomous differential equations of first order. Throughout it is assumed that the ω_ν are polynomials in $\boldsymbol{x}$. The infinitesimal generator U of a point symmetry is defined by

$$U=\xi(t,\boldsymbol{x})\,\partial/\partial t+\eta_\alpha(t,\boldsymbol{x})\,\partial/\partial x_\alpha. \tag{2}$$

Summation from 1 to n over twice occurring indices is always understood. The infinitesimals ξ and η_α have to be determined from the invariance condition $U^{(1)}\Omega_\nu=0$ on $\Omega_\nu=0$ where $U^{(1)}$ is the first extension of U. The following system of linear partial differential equations for ξ and the η_α's is obtained

$$\partial\eta_\nu/\partial t+\omega_\alpha\,\partial\eta_\nu/\partial x_\alpha-\eta_\alpha\,\partial\omega_\nu/\partial x_\alpha-\omega_\nu(\partial\xi/\partial t+\omega_\alpha\,\partial\xi/\partial x_\alpha)=0. \tag{3}$$

The general solution of the system (3) determines the full symmetry group for equations (1). This is in most cases impossible.

Symmetries often make it possible to diminish the number of dependent variables. To this end the variables $\boldsymbol{x}$ have to be transformed such that the corresponding infinitesimal generator of the symmetry becomes a translation in one of the dependent variables. From this it is clear that only those symmetries are useful for the integration of (1) for which this transformation may actually be performed.

Let us now discuss several special solutions of (3) from this point of view:

(i) For arbitrary ω_α there exist the obvious solutions

$$\xi=\text{constant},\qquad \eta_\nu=0,\qquad t\text{ translations} \tag{4a}$$

$$\xi=f(t,\boldsymbol{x}),\qquad \eta_\nu=f(t,\boldsymbol{x})\omega_\nu,\qquad f\text{ arbitrary}. \tag{4b}$$

These symmetries are useless for the integration of (1) for the reason mentioned above.

(ii) If $\eta_\nu = 0$ for all ν and ξ does not depend on t, then the system (3) reduces to

$$\omega_\alpha \, \partial\xi(\boldsymbol{x})/\partial x_\alpha = 0. \tag{5}$$

If $F(\boldsymbol{x})$ is a solution of (5) it is a time independent first integral and the system (1) allows the symmetry generator

$$U = F(\boldsymbol{x}) \, \partial/\partial t. \tag{6}$$

Symmetries of this kind are the most useful ones for the integration.

(iii) Again let $\eta_\nu = 0$ for all ν and an exponential time dependence for ξ be assumed, i.e.

$$\xi(t, \boldsymbol{x}) = \mathrm{e}^{kt}\xi_0(\boldsymbol{x}). \tag{7}$$

The system (3) reduces to

$$k\xi_0(\boldsymbol{x}) + \omega_\alpha \, \partial\xi_0(\boldsymbol{x})/\partial x_\alpha = 0. \tag{8}$$

The solutions of (7) and (8) are the time dependent first integrals which have been proposed by one of the authors (Steeb 1982). If such an integral $F(t, \boldsymbol{x})$ with exponential time dependency can be found, the system (1) again allows a symmetry generator of the form (6) with a time dependent function F.

More general symmetry generators may be obtained if no restrictions on ξ are imposed. This discussion shows the relation between time independent first integrals, time dependent first integrals with exponential time dependence and a general symmetry generator.

We now consider several autonomous systems of the form (1) and determine a certain class of their symmetries. It turns out that all calculations in this field are very lengthy. For this reason there have been developed two computer algebra packages (Schwarz 1984a, b). If polynomial dependency of a predefined degree in $\boldsymbol{x}$ is assumed, they determine the time independent and the time dependent first integrals and more general infinitesimal generators of a symmetry almost completely automatically. Most results described below have been obtained by applying these REDUCE packages.

We begin with the Lotka–Volterra equations for three competing populations which has been considered by Steeb and Erig (1983) and Steeb *et al* (1983), namely

$$\dot{x}_1 = x_1(1 + ax_2 + bx_3), \qquad \dot{x}_2 = x_2(1 - ax_1 + bx_3), \qquad \dot{x}_3 = x_3(1 - bx_1 - cx_2) \tag{9}$$

where a, b and c are real parameters. For arbitrary a, b, c there is the first integral

$$I(t, \boldsymbol{x}) = (x_1 + x_2 + x_3)\,\mathrm{e}^{-3t}. \tag{10}$$

In addition there is the integral

$$I(t, \boldsymbol{x}) = x_1^k x_2^l x_3^m \,\mathrm{e}^{-(k+l+m)t} \tag{11}$$

if k, l and m are determined from $ak - cm = 0$, $bk + cl = 0$. For special values of the parameters there exist time dependent integrals of higher order in $\boldsymbol{x}$. If $b = -a$, $c = a$ there is the third-order integral

$$I(t, \boldsymbol{x}) = [x_1^3 + x_2^3 + x_3^3 + 3x_1(x_2^2 + x_3^2) + 3x_2(x_1^2 + x_3^2) + 3x_3(x_1^2 + x_2^2)]\,\mathrm{e}^{-3t}. \tag{12}$$

If $b = -a$, $c = 2a$ there is the fourth-order integral

$$\begin{aligned} I(t, \boldsymbol{x}) = {} & [x_1^4 + x_2^4 + x_3^4 + 4x_1(x_2^3 + x_3^3) + 4x_2(x_1^3 + x_3^3) + 4x_3(x_1^3 + x_2^3) \\ & + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 12x_1x_2x_3(x_2 + x_3)]\,\mathrm{e}^{-4t}. \end{aligned} \tag{13}$$

There exist two more integrals of this type for suitably restricted coefficients a, b and c which are fifth-order polynomials in x. It is suggested that there is a hierarchy of polynomial first integrals. Concerning the existence of more general symmetries with polynomial dependency in x, there is no solution up to degree three. Non-polynomial first integrals have been determined by Steeb and Erig (1983).

Chemical reactions are typically described by autonomous systems of the form (1). The dependent variables describe how the concentrations of the various chemicals evolve in time. A reaction derived from Selkov's model (Selkov 1968) has been studied by Richter (1984). The system is

$$\dot{x}_1 = a - cx_1 - x_1x_2^2 + dx_2^3, \qquad \dot{x}_2 = b - x_2 + x_1x_2^2 - dx_2^3. \tag{14}$$

Up to fifth order in the dependent variables x there is only the time dependent first integral

$$I(t, x) = (x_1 + x_2 - a - b)\, e^t \tag{15}$$

if $c = 1$. These parameters are not in the range covered by the plot given in figure 1 of Richter (1984). Furthermore, there is no more general symmetry generator with polynomial dependence of order less than three on the x_α's.

There is another chemical reaction which also only allows for a single time dependent first integral. It is the interaction between two stirred cell reactors either of which contains two chemicals reacting with each other (Holmes and Moon 1983). It is described by the fourth-order system

$$\begin{aligned} &\dot{x}_1 = a - (b+1)x_1 + x_1^2x_3 - c(x_1 - x_2), \qquad &&\dot{x}_3 = bx_1 - x_1^2x_3 - d(x_3 - x_4) \\ &\dot{x}_2 = a - (b+1)x_2 + x_2^2x_4 + c(x_1 - x_2), \qquad &&\dot{x}_4 = bx_2 - x_2^2x_4 + d(x_3 - x_4). \end{aligned} \tag{16}$$

Up to third order in x there is only the first integral

$$I(t, x) = (x_1 - x_2 + x_3 - x_4)\, e^{(2c+1)t} \tag{17}$$

if the parameters are constrained by $d = c + \frac{1}{2}$.

The following system

$$\dot{x}_1 = -x_2 - x_3, \qquad \dot{x}_2 = x_1 + ax_2, \qquad \dot{x}_3 = b + x_1x_3 - cx_3 \tag{18}$$

has been studied by Rössler (1976) as a model for a chemical reaction. There is neither a first integral nor a more general symmetry generator with polynomial dependency on x of order lower than six.

Next consider the fourth-order system

$$\dot{x}_1 = -ax_2, \qquad \dot{x}_2 = x_3x_4 + ax_1, \qquad \dot{x}_3 = x_2x_4, \qquad \dot{x}_4 = -x_2. \tag{19}$$

It arises by symmetry reduction (space–time translations) of the reduced Maxwell–Bloch system. Since the ω_ν in system (18) do not depend on x_ν, time independent first integrals are expected. Indeed the following three polynomial first integrals are obtained

$$I_1(x) = x_1 - ax_4, \qquad I_2(x) = x_4^2 - 2x_3, \qquad I_3(x) = x_1^2 + x_2^2 + x_3^2. \tag{20}$$

With the help of these first integrals the general solution of the system (19) may be obtained explicitly. In this case there also exist more general symmetry generators with polynomial infinitesimals ξ and η. If the order of the polynomials is restricted to two there are two infinitesimal generators in addition to those specified under (i) which always exist. They are

$$U_1 = -t\,\partial/\partial t + (3x_1 - 2ax_4)\,\partial/\partial x_1 + 2x_2\,\partial/\partial x_2 + 2(x_3 + a^2)\,\partial/\partial x_3 + x_4\,\partial/\partial x_4$$

$$U_2 = (x_2^2 + x_3^2 + a^2x_3 + 2ax_1x_4 - 2a^2x_4^2)\,\partial/\partial x_1 - ax_2x_4\,\partial/\partial x_2 - a(3ax_1 - 2a^2x_4 + x_3x_4)\,\partial/\partial x_3 - a(x_3 + a^2)\,\partial/\partial x_4. \tag{21}$$

Together with the vector field V which is defined by the right-hand sides of (19) they form a three-dimensional subalgebra in the full symmetry algebra of the system (19) obeying the commutation relations $[U_1, U_2] = U_2$, $[U_1, V] = V$ and $[U_2, V] = 0$, U_1 is the generator of scaling symmetry. The generators of the form (6) defined by the integrals (20) are an invariant algebra with respect to $U_{1,2}$.

There are two more models which we have investigated; the famous Lorenz model

$$\dot{x}_1 = \sigma(x_2 - x_1), \qquad \dot{x}_2 = -x_1x_3 + rx_1 - x_2, \qquad \dot{x}_3 = x_1x_2 - bx_3, \tag{22}$$

and the Rikitake two-disc dynamo system

$$\dot{x}_1 = -\mu x_1 + x_2x_3, \qquad \dot{x}_2 = -\mu x_2 - \alpha x_1 + x_1x_3, \qquad \dot{x}_3 = 1 - x_1x_2. \tag{23}$$

For both systems we do not find any first integrals besides those already known in the literature. However, since most of our calculations are done automatically the range of validity of these statements has been enlarged considerably. For both systems (22) and (23) there does not exist an additional time dependent first integral with polynomial dependency on x of order lower than six. Furthermore there is no other symmetry generator with polynomial dependency on t and x of order lower than three.

A comment is in order about the singular point analysis (Ablowitz *et al* 1980) and its connection with the existence of first integrals of autonomous systems. The singular point analysis gives a hint for finding first integrals of autonomous systems. The singular point analysis gives a hint for finding first integrals (Tabor and Weiss 1981, Steeb *et al* 1983). In particular polynomial first integrals are obtained when the autonomous system (1) has the Painlevé property, where the ω_α's are polynomial. Recently Yoshida (1983a, b) obtained a necessary condition for the existence of algebraic first integrals originating from exact or approximate symmetry of the autonomous system under similarity transformations. In particular he is able to derive the resonances (he calls them Kowalevski exponents) from the knowledge of the dominant behaviour of the system of ordinary differential equations and the polynomial first integrals (if any exist). Let us explain Yoshida's ideas with the help of equation (19) (considered in the complex domain). Inserting

$$x_i(t) \sim a_i(t - t_0)^{n_i} \tag{24}$$

yields $n_1 = n_4 = -1$, $n_2 = n_3 = -2$ and $a_i \neq 0$ for all i. Retaining only the leading terms the equations are

$$\dot{x}_1 = -ax_2, \qquad \dot{x}_2 = x_3x_4, \qquad \dot{x}_3 = -x_2x_4, \qquad \dot{x}_4 = -x_2 \tag{25}$$

with the first integrals

$$I_1(x) = x_1 - ax_4, \qquad I_2(x) = x_4^2 - 2x_3, \qquad I_3(x) = x_2^2 + x_3^2. \tag{26}$$

Due to Yoshida the first integrals I_1 and I_2 are associated with the resonances $r = 1$ and $r = 2$ because $I_1(\alpha x_1, \alpha x_2) = \alpha I_1(x_1, x_2)$ and $I_2(\alpha^2 x_3, \alpha x_4) = \alpha^2 I_2(x_3, x_4)$ respectively; x_i is multiplied by α to the power $-n_i$. The exponents of α at the right-hand sides are the resonances. From the remaining first integral I_3 the resonance $r = 4$ is obtained since

$$I_3(\alpha^2 x_2, \alpha^2 x_3) = (\alpha^2 x_2^2)^2 + (\alpha^2 x_3^2)^2 = \alpha^4(x_2^2 + x_3^2). \tag{27}$$

It is one of the major achievements of Yoshida to clarify the connection between invariance properties of a system of autonomous equations and the singular point analysis.

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